Generation of Steiner Triple Systems of order 21 by CoDA

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Abstract

CoDA is a software system for construction and study of *Combinatorial Designs* with predefined *Automorphisms*. This example shows the way it can be used when solving a definite construction or classification problem.

1 Introduction

Design Theory is one of the most rapidly developed branches of modern Combinatorics. Designs are interesting combinatorial configurations that are often used in the construction, study and application of some codes and graphs ([1], [10], [2]) as well as in the statistical planning of experiments [8].

Let $V = \{P_i\}_{i=1}^{v}$ be a finite set of *points*, and $\mathcal{B} = \{B_j\}_{j=1}^{b}$ – a finite collection of k-element subsets of V, called *blocks*. We say that $D = (V, \mathcal{B})$ is a *design* with parameters t-(v,k,λ), if any t-subset of V is contained in exactly λ blocks of $\mathcal{B}$.

The incidence matrix of a design is a (0,1) matrix of v rows and b columns, where the element of the i-th row and j-th column equals 1, if $P_i \in B_j$ $(i = 1, 2, ..., v,\ j = 1, 2, ..., b)$ and 0 if not. The design is completely determined by its incidence matrix.

Two designs are *isomorphic* if the incidence matrix of the first design can be obtained from the incidence matrix of the second one by changing the places of its rows and columns.

An *automorphism* is an isomorphism of the design to itself, i.e. a permutation of the points that transforms the blocks into blocks. All the automorphisms of a design form a group called its *full group of automorphisms*. Each subgroup of this group is a group of automorphisms of the design.

Design Theory deals with the problems of the existence, enumeration and classification of designs with certain parameters, as well as with the study of their properties, and their relationship to other combinatorial configurations.

With the growth of the parameters, the classification of all nonisomorphic designs becomes impossible. That is why a classification of only some of the designs is been often done. In many cases the designs with certain automorphisms (additional symmetry) are classified.

One of the most popular methods for construction of combinatorial designs with some automorphisms is the so called *local approach*. It has been successfully used for several decades by plenty of scientists - Anstee, Hall, Thompson, Van Trung, Mathon, Rosa, Phelps, Spense, Janko, Abel, Mills, Becker, Piper, Tonchev, Kapralov, Landjev ... The list is too long to be completed, and it is impossible to include an extended bibliography here, but citings can be found in [2], [10], [1] and in many other monographs on Design Theory.

There always exist automorphisms of prime orders which generate the full group of automorphisms of a design. Let us consider a design possessing an automorphism α of a prime order p with f fixed points and h fixed blocks. It can be transformed into an isomorphic design such that α acts on the points as

$$(1)(2)...(f)(f{+}1, f{+}2, ..., f{+}p)(f{+}p{+}1, f{+}p{+}2, ..., f{+}2p)...(f{+}lp{+}1, f{+}lp{+}2, ..., f{+}(l{+}1)p)$$

and on the blocks as

$$(1)(2)...(h)(h{+}1, h{+}2, ..., h{+}p)(h{+}p{+}1, h{+}p{+}2, ..., h{+}2p)...(h{+}mp{+}1, h{+}mp{+}2, ..., h{+}(m{+}1)p).$$

Let us consider the incidence matrix of this design. The part of it which corresponds to the nontrivial point and block orbits consists of circulant matrices of order p. This specific presentation of the incidence matrix by means of circulant matrices of order p is used by the *local approach* to make the construction easier. It is called *local* because we just choose one of all the theoretically possible automorphisms and consider the specific picture of its incidence matrix split into circulants. We then construct designs with the predefined automorphism group.

CoDA is a set of programmes for construction of designs by this method, and for their classification with respect to the order of the automorphism group and other properties.

2 The Problem

A Steiner triple system of order v (denoted STS(v)) is a 2-(v,3,1) design (see for instance [3]). Steiner triple systems of orders $v < 19$ have been classified quite long ago. Recently Ostergard and Kaski completed the classification of STS(19) [5], so that STS(21) are now the Steiner triple systems with smallest parameters that have still not been classified, or at least enumerated.

Wilson showed in 1974 that the nonisomorphic 2-(21,3,1) designs with three subsystems STS(7) are at least 2160980 [15]. Classifications of some other smaller classes of STS(21) have been done by several authors, i.e. Mathon, Phelps and Rosa [6], [7], Tonchev [9], [10], Kapralov and Topalova [4]. The exact number of the nonisomorphic Wilson type STS(21) was determined by Topalova and Zlatarski, who also classified them with respect to the order of the automorphism groups. The general approach will be briefly described here. As far as the solution of the problem was split into several relatively independent steps, it will only be illustrated in detail how CoDA was used at one of these steps.

The whole number of designs resulting from Wilson's construction (including the isomorphic ones) should be $30^3 7!6!16942080$ [15]. A design with no automorphisms has $3!(7!)^3$ isomorphic ones among them, so Wilson concludes that there are at least

$$\frac{30^3 7!6!16942080}{3!(7!)^3} = 2160980$$

nonisomorphic STS(21).

If we know the number of all the Wilson type STS(21) with nontrivial automorphisms, we can determine the exact number of these designs using the more precise formula

$$\sum_{aut} \frac{N_{aut} 3!(7!)^3}{aut} = 30^3 7!6!16942080.$$

Here N_{aut} denotes the number of designs with a full automorphism group of order aut.

It was proved [14] that a Wilson type STS(21) can possess the following automorphisms of prime orders:

1. An automorphism of order 2 with 9 fixed points and 18 fixed blocks.
2. An automorphism of order 2 with 7 fixed points and 14 fixed blocks.
3. An automorphism of order 2 with 3 fixed points and 10 fixed blocks.
4. An automorphism of order 3 with 3 fixed points and 4 fixed blocks.
5. An automorphism of order 3 with no fixed points and 1, 3, or 7 fixed blocks.
6. An automorphism of order 7 with 7 fixed points and 7 fixed blocks.
7. An automorphism of order 7 with no fixed points and no fixed blocks.

The different possible automorphism cases were attacked one by one [11], [12], [13] in different ways, and this way the problem was solved for the first five automorphism cases. All the designs with automorphisms of order 7 (items 6. and 7.) had already been constructed [9], but it was not known which of them are of Wilson type, and which not, so we had to check them all. But not all known designs have been made by the authors accessible at some web-page. Thus it is not fast to rewrite them from old papers, and while doing it, quite many errors are possible. On the other hand generating these designs by CoDA is a fast and easy job. It is discussed in detail in the next section.

3 Wilson STS(21) with automorphisms of order 7

Let D be a 2-(21,3,1) design with an automorphism φ of order 7 fixing no points and no blocks. Without loss of generality we can assume that φ acts as follows:
φ=(1,2,3,4,5,6,7)(8,9,10,11,12,13,14)(15,16,17,18,19,20,21) on the points,
φ=(1,2,3,4,5,6,7)(8,9,10,11,12,13,14)...(64,65,66,67,68,69,70) on the blocks. Then the incidence matrix of D is of the form:

$$\begin{pmatrix} B_{1,1} & B_{1,2} & B_{1,3} & B_{1,4} & B_{1,5} & B_{1,6} & B_{1,7} & B_{1,8} & B_{1,9} & B_{1,10} \\ B_{2,1} & B_{2,2} & B_{2,3} & B_{2,4} & B_{2,5} & B_{2,6} & B_{2,7} & B_{2,8} & B_{2,9} & B_{2,10} \\ B_{3,1} & B_{3,2} & B_{3,3} & B_{3,4} & B_{3,5} & B_{3,6} & B_{3,7} & B_{3,8} & B_{3,9} & B_{3,10} \end{pmatrix}$$

where $B_{i,j}$, $i = 1,2,3$, $j = 1,2,\ldots,10$ are circulant matrices of order 7.

Let $n_{i,j}$, $i = 1,2,3$, $j = 1,2,\ldots,10$ be equal to the number of 1's in a row of $B_{i,j}$.The following equations hold for the matrix $N = (n_{i,j})_{3\mathrm{X}10}$

$$\sum_{j=1}^{10} n_{i,j} = 10, \qquad i = 1,2,3 \tag{1}$$

$$\sum_{j=1}^{10} n_{i,j}^2 = 16, \qquad i = 1,2,3 \tag{2}$$

$$\sum_{j=1}^{10} n_{i_1,j} n_{i_2,j} = 7, \qquad 1 \le i_1 < i_2 \le 3. \tag{3}$$

A Wilson STS(21) contains three subsystems STS(21), and thus N should contain three columns of type (3,0,0). It follows that up to equivalence there is only one possible solution for N, i.e.

$$N$$

$$\begin{matrix} 3&0&0&1&1&1&1&1&1&1 \\ 0&3&0&1&1&1&1&1&1&1 \\ 0&0&3&1&1&1&1&1&1&1 \end{matrix}$$

We are interested in extending it to designs by CoDA. That is why we prepare a file with information about the problem, and the matrix N. The file contains:

```
file o7f0.dat

1   3 0 0 1 1 1 1 1 1 1

1   0 3 0 1 1 1 1 1 1 1

1   0 0 3 1 1 1 1 1 1 1


      1   2   3   4   5   6   7   8   9  10  11  12  13  14  15  16  17  18  19  20  21


 1   10   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1   1
 8    1   1   1   1   1   1   1  10   1   1   1   1   1   1   1   1   1   1   1   1   1
15    1   1   1   1   1   1   1   1   1   1   1   1   1   1  10   1   1   1   1   1   1
```

```
    1  1

    0 0 0 0 0 0 0 0 0 0
    0 0 0 0 0 0 0 0 0 0
    0 0 0 0 0 0 0 0 0 0


    0 0



legend: orbit matrix rows:
        type of the row, the row itself

        lambda of the rows

        which i j we start from
        the design itself up to now
```

We give the file $o7f0.dat$ as parameter to the program for extension of orbit matrices to designs, and as a result it writes in the same file

```
 D1
15  0  0  1  1  1  1  1  1  1
 0 15  0  1  2  3  4  5  6  7
 0  0 15  1  3  5  7  2  4  6


...


 D47
15  0  0  1  1  1  1  1  1  1
 0 15  0  1  2  3  4  5  6  7
 0  0 22  6  4  2  1  5  7  3



 All:47
```

The designs are written in short - each integer in the matrix is the number of a circulant. These numbers are specified in the file circnum.dat. Thus it is easy to construct the designs from the abbreviations above. We then give the file with the designs as a parameter to the program checking for design isomorphism. As a result only 6 nonisomorphic designs remain.

We next study the automorphism groups of these designs. As input to the program calculating the automorhism group we give the name of the file with the designs, and of an output file where the results will be written. As a result this output file will contain the order and the generating permutations of the automorphism group of each design, i.e.

```
d1-D1
(1,2,3,4,5,6,7)(8)(9)(10)(11)(12)(13)(14)(15,21,20,19,18,17,16)
 2 3 4 5 6 7 1 8 9 10 11 12 13 14 21 15 16 17 18 19 20
(1,15)(2,16)(3,17)(4,18)(5,19)(6,20)(7,21)(8)(9)(10)(11)(12)(13)(14)
 15 16 17 18 19 20 21 8 9 10 11 12 13 14 1 2 3 4 5 6 7
A1 = 14
(1)(2,3,5)(4,7,6)(8)(9,10,12)(11,14,13)(15)(16,17,19)(18,21,20)
 1 3 5 7 2 4 6 8 10 12 14 9 11 13 15 17 19 21 16 18 20
A2 = 3
(1)(2)(3)(4)(5)(6)(7)(8,9,10,11,12,13,14)(15,17,19,21,16,18,20)
 1 2 3 4 5 6 7 9 10 11 12 13 14 8 17 18 19 20 21 15 16
A8 = 7
Aut = 294


d2-21 42 D2
(1,2,3,4,5,6,7)(8,9,10,11,12,13,14)(15,16,17,18,19,20,21)
 2 3 4 5 6 7 1 9 10 11 12 13 14 8 16 17 18 19 20 21 15
(1,8,7,14,6,13,5,12,4,11,3,10,2,9)(15,18,21,17,20,16,19)
 8 9 10 11 12 13 14 7 1 2 3 4 5 6 18 19 20 21 15 16 17
(1,15,14,4,18,10,7,21,13,3,17,9,6,20,12,2,16,8,5,19,11)
 15 16 17 18 19 20 21 5 6 7 1 2 3 4 14 8 9 10 11 12 13
A1 = 21
(1)(2,3,5)(4,7,6)(8,11,10)(9,13,14)(12)(15,16,18)(17,20,19)(21)
 1 3 5 7 2 4 6 11 13 8 10 12 14 9 16 18 20 15 17 19 21
A2 = 3
(1)(2)(3)(4)(5)(6)(7)(8,17)(9,18)(10,19)(11,20)(12,21)(13,15)(14,16)
 1 2 3 4 5 6 7 17 18 19 20 21 15 16 13 14 8 9 10 11 12
A8 = 2
Aut = 126
```

```
d3-21 42 D3
(1,2)(3)(4,7)(5)(6)(8)(9,13)(10,11)(12)(14)(15,21)(16)(17,20)(18)(19)
 2 1 3 7 5 6 4 8 13 11 10 12 9 14 21 16 20 18 19 17 15
(1,3,2)(4,7,6)(5)(8,13,9)(10,12,11)(14)(15,21,16)(17,20,19)(18)
 3 1 2 7 5 4 6 13 8 12 10 11 9 14 21 15 20 18 17 19 16
(1,4,5,7,6,3,2)(8,12,11,10,13,14,9)(15,21,17,18,20,19,16)
 4 1 2 5 7 3 6 12 8 13 10 11 14 9 21 15 18 20 16 19 17
(1,8,4,11,7,14,3,10,6,13,2,9,5,12)(15,20,18,16,21,19,17)
 8 9 10 11 12 13 14 4 5 6 7 1 2 3 20 21 15 16 17 18 19
(1,15,12,4,18,8,7,21,11,3,17,14,6,20,10,2,16,13,5,19,9)
 15 16 17 18 19 20 21 7 1 2 3 4 5 6 12 13 14 8 9 10 11
A1 = 21
(1)(2,3)(4,6)(5)(7)(8,13)(9)(10)(11,12)(14)(15,16)(17,19)(18)(20)(21)
 1 3 2 6 5 4 7 13 9 10 12 11 8 14 16 15 19 18 17 20 21
(1)(2,4,6,3)(5,7)(8,12,11,13)(9,14)(10)(15,17,19,16)(18,20)(21)
 1 4 2 6 7 3 5 12 14 10 13 11 8 9 17 15 19 20 16 18 21
(1)(2,5,3)(4,6,7)(8,9,13)(10)(11,14,12)(15,18,16)(17,19,20)(21)
 1 5 2 6 3 7 4 9 13 10 14 11 8 12 18 15 19 16 20 17 21
A2 = 6
(1)(2)(3,4)(5,7)(6)(8)(9,14)(10)(11)(12,13)(15)(16,17)(18,20)(19)(21)
 1 2 4 3 7 6 5 8 14 10 11 13 12 9 15 17 16 20 19 18 21
(1)(2)(3,5)(4,7)(6)(8)(9,13)(10)(11)(12,14)(15)(16,18)(17,20)(19)(21)
 1 2 5 7 3 6 4 8 13 10 11 14 9 12 15 18 20 16 19 17 21
A3 = 4
(1)(2)(3)(4)(5)(6)(7)(8,19)(9,20)(10,21)(11,15)(12,16)(13,17)(14,18)
 1 2 3 4 5 6 7 19 20 21 15 16 17 18 11 12 13 14 8 9 10
A8 = 2
Aut = 1008
```

```
d4-21 42 D5
(1,2,3,4,5,6,7)(8)(9)(10)(11)(12)(13)(14)(15,20,18,16,21,19,17)
 2 3 4 5 6 7 1 8 9 10 11 12 13 14 20 21 15 16 17 18 19
(1,8)(2,9,5,12,3,10)(4,11,6,13,7,14)(15)(16,17,19)(18,21,20)
 8 9 10 11 12 13 14 1 5 2 6 3 7 4 15 17 19 21 16 18 20
(1,15,8)(2,16,9)(3,17,10)(4,18,11)(5,19,12)(6,20,13)(7,21,14)
 15 16 17 18 19 20 21 1 2 3 4 5 6 7 8 9 10 11 12 13 14
A1 = 21
```

```
(1)(2,3,5)(4,7,6)(8)(9,10,12)(11,14,13)(15)(16,17,19)(18,21,20)
 1 3 5 7 2 4 6 8 10 12 14 9 11 13 15 17 19 21 16 18 20
A2 = 3
(1)(2)(3)(4)(5)(6)(7)(8,9,10,11,12,13,14)(15,18,21,17,20,16,19)
 1 2 3 4 5 6 7 9 10 11 12 13 14 8 18 19 20 21 15 16 17
(1)(2)(3)(4)(5)(6)(7)(8,15)(9,19)(10,16)(11,20)(12,17)(13,21)(14,18)
 1 2 3 4 5 6 7 15 19 16 20 17 21 18 8 10 12 14 9 11 13
A8 = 14
Aut = 882


d5-21 42 D20
(1,2,3,4,5,6,7)(8,9,10,11,12,13,14)(15,16,17,18,19,20,21)
 2 3 4 5 6 7 1 9 10 11 12 13 14 8 16 17 18 19 20 21 15
(1,8,7,14,6,13,5,12,4,11,3,10,2,9)(15,18,21,17,20,16,19)
 8 9 10 11 12 13 14 7 1 2 3 4 5 6 18 19 20 21 15 16 17
A1 = 14
(1)(2,3,5)(4,7,6)(8,11,10)(9,13,14)(12)(15,16,18)(17,20,19)(21)
 1 3 5 7 2 4 6 11 13 8 10 12 14 9 16 18 20 15 17 19 21
A2 = 3
Aut = 42


d6-21 42 D21
(1,2,3,4,5,6,7)(8,9,10,11,12,13,14)(15,16,17,18,19,20,21)
 2 3 4 5 6 7 1 9 10 11 12 13 14 8 16 17 18 19 20 21 15
(1,8,4,11,7,14,3,10,6,13,2,9,5,12)(15,20,18,16,21,19,17)
 8 9 10 11 12 13 14 4 5 6 7 1 2 3 20 21 15 16 17 18 19
A1 = 14
(1)(2,3,5)(4,7,6)(8,13,9)(10)(11,12,14)(15,16,18)(17,20,19)(21)
 1 3 5 7 2 4 6 13 8 10 12 14 9 11 16 18 20 15 17 19 21
A2 = 3
Aut = 42
```

The generating permutations are presented in two ways - as cycles, and as the new arrangement of the points of the design. *Aut* denotes the order of the full automorphism group.

In the example above we can see that all these designs possess not only the automorphism of order 7 we supposed, but also some other automorphisms. Yet we have already constructed all the designs with other possible automorphisms, i.e. we already have these designs.

It is easy to see that an STS(21) with an automorphism of order 7 with 7 fixed points [9] is always of Wilson type. Their automorphism group orders are always greater than 7, so we have already constructed them too.

4 Results

The consideration of Wilson STS(21) with automorphisms of order 7 was the last step in the classification of all the Wilson STS(21). The final results are summarized in Table 1.

Table 1: Automorphism group orders of Wilson type designs

Aut.Gr.Order	1	2	3	4	6	8	9	12	16	18	
Designs	2156186	8914	685	271	197	41	3	6	4	7	
Aut.Gr.Order	21	24	42	48	72	126	144	294	882	1008	All
Designs	2	17	6	2	5	1	1	1	1	1	2166351

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